Reaction Solvability Analysis using Natural Coordinates

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Abstract. In over-constrained mechanisms, all the joint reactions cannot be solved uniquely based solely on rigid body assumptions. However, a few joint reactions may be uniquely solvable, and an approach termed as Reaction Solvability Analysis (RSA) in this paper, can be used to find such uniquely solvable joint reactions. Existing work have implemented RSA algorithms using absolute coordinates. In this work, the RSA algorithm is used with natural coordinates and this is found to be more efficient for finding uniquely solvable joint reactions. To use natural coordinates for RSA, they need to be modified and this is discussed in this work.

Keywords: Reaction Solvability Analysis · Overconstrained Mechanisms · Natural Coordinates.

1 Introduction

Over-constrained mechanisms have actual degree of freedom (DOF) more than the number computed using the well-known Grübler-Kutzbach criterion[1]. This happens because of the presence of multiple joints, called redundant joints, constraining the same degrees of freedom. These redundant constraints cause linear dependency in constraint equations, leading to singular Jacobian matrices. The redundant constraints can be removed arbitrarily, without changing its kinematics [2], and the kinematics of the system can be solved. However, one can not solve for joint reactions uniquely in such a system solely on the basis of rigid body constraint equations, and one needs to include fexibility/the material constitutive equations to obtain the joint reactions. This issue of "*joint reaction indeterminacy*" is the topic of this work.

The usual approach for handling redundant constraints is to solve for the system by eliminating any arbitrary set of dependent constraints equations [3, 4], or by using algorithms capable of dealing with dependent equations (e.g. minimum-norm solution and augmented Lagrangian) [5, 6, 7], or by using penalty-based and weighing factors based methods[8, 9]. However, these approaches are good for kinematic and certain dynamic analyses, and do not yield correct results for joint reactions. For calculating joint reactions, one needs to discard the rigid body assumption, introduce material constitutive relations to have the complete set of equations, and solve the system using a finite element approach (FEA) or analytically[10, 11, 12].

A new strategy was suggested based on the observation that in an over-constrained mechanism, *some joint reactions may be solvable* without considering the material constitutive equations[10]. Hence, if our joint reactions of interest lie in this set of "solvable joint reactions", then there is no need to use an FEA solver. This can save tremendously in computation for complex mechanisms. In the subsequent works [10, 2, 5], algorithms were developed to find such "solvable joint reactions" in over-constrained mechanisms. This analysis for solvable joint reactions is termed as "Reaction Solvability Analysis (RSA)".

In existing works, the coordinate system of choice for RSA has been absolute coordinates. In this work, we show that using natural coordinates can lead to substantial benefits in terms of computational simplicity and efficiency. The main reason is that the constraint equations using absolute coordinates are transcendental, while the constraint equations in natural coordinates are maximally quadratic, leading to a Jacobian matrix with linear terms. However, the application of natural coordinates for RSA is not straightforward, and we had to modify the natural coordinates to make them usable for RSA. We demonstrate the RSA application using our modified natural coordinates with a planar over-constrained mechanism.

2 Equations of Motion and the RSA Algorithm

In this section, we discuss the necessary equations and representation that we have used in this paper, and the existing RSA algorithms that have been developed in prior works.

2.1 Equations of Motion of Multi-body Systems

We will be using the standard representation used in [13] and [14] to represent the dynamics and constraint equations of multi-body systems.

The equations of motion of a multi-body system can be written in a compact form as

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{f} = \mathbf{Q} \tag{1}$$

where $\mathbf{M_{nxn}}$ is the mass/inertia matrix, $\mathbf{q_{nx1}} = (q_1, q_2, ..., q_n)^T$ is the vector of generalized coordinates, and $\mathbf{Q_{nx1}}$ is the vector of external forces and other inertia terms such as the Coriolis and centripetal terms. The constraint equations of a multi-body system can be written as

$$\boldsymbol{\Phi}(\mathbf{q}) = \begin{bmatrix} \Phi_1(\mathbf{q}) \\ \Phi_2(\mathbf{q}) \\ \vdots \\ \Phi_m(\mathbf{q}) \end{bmatrix} = \mathbf{0} \tag{2}$$

where, $\mathbf{\Phi}(\mathbf{q}) : R^n \rightarrow R^m$ is the vector containing all scalar constraint equations.

If some of the equations in (2) are dependent, it gives rise to a redundantly constrained system. It is convenient to check for this redundancy by checking the rank of the Jacobian matrix of this system where the $m \times n$ Jacobian matrix of this system can be written as

$$\mathbf{\Phi_q}(\mathbf{q}) = \begin{pmatrix} \frac{\partial \Phi_1}{\partial q_1} & \frac{\partial \Phi_1}{\partial q_2} & \cdots & \cdots \\ \frac{\partial \Phi_2}{\partial q_1} & \frac{\partial \Phi_2}{\partial q_2} & \cdots & \cdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \Phi_m}{\partial q_1} & \frac{\partial \Phi_m}{\partial q_2} & \cdots & \frac{\partial \Phi_m}{\partial q_n} \end{pmatrix} \tag{3}$$

The generalised force vector $f_{n\times1}$ can be written as

$$\boxed{\mathbf{f} = \mathbf{\Phi_q}^T \lambda} \tag{4}$$

where $\lambda_{m\times1}$ is the vector of Lagrange multipliers. Equation (4) is used to obtain constraint reaction forces from the given constraint equations.

2.2 The RSA Algorithm

From equation (1) and (4), we can write

$$\mathbf{\Phi_q}^T \lambda = \mathbf{Q} - \mathbf{M}\ddot{\mathbf{q}} = \mathbf{f} \tag{5}$$

The above equation (5) is a convenient representation in the familiar linear-algebraic form $Ax = b$. Based on the analysis of this equation , a simple RSA algorithm was proposed[10]. In this algorithm, first we split the Jacobian matrix $\mathbf{\Phi_q}$ into two matrices – one, which contains constraints acting on a particular joint and other which contains constraints **not** acting on that particular joint. To analyze the joint **i**, we can get two matrices for that joint as

$$\mathbf{\Phi_q^i} : \text{Matrix with all the constraints (rows) acting on joint } \mathbf{i}$$
$$\mathbf{\Phi_q^{-i}} : \text{Matrix with all the constraints (rows) not acting on joint } \mathbf{i}$$

We can now find the ranks of the corresponding matrices as

$$r_i : \text{Rank of Matrix } \mathbf{\Phi_q^i}$$
$$r_{-i} : \text{Rank of Matrix } \mathbf{\Phi_q^{-i}}$$

The algorithm states that the constraints forces for joints can be solved for uniquely for which the following relation holds:

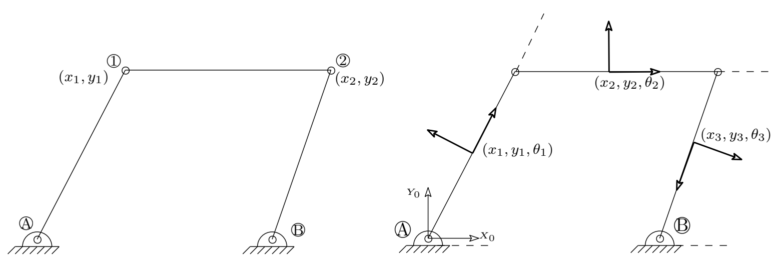
$$\boxed{r = r_i + r_{-i}} \tag{6}$$

3 The Optimal Coordinate Formulation for RSA

To implement the RSA algorithm discussed in the previous section, one needs a formulation **which has constraint equations for each joint**. This is the key idea differentiating which coordinates are "usable" for RSA and which are not.

The two commonly used types of coordinates are the absolute coordinates (or reference point coordinates) and natural coordinates (or fully Cartesian coordinates). In absolute coordinates, a coordinate frame is attached to every link, and joints are defined by the constraint equations between these frames. Since the constraint equations are for joints, absolute coordinates are directly usable for RSA and have been used in prior works. In natural coordinates, one defines points and unit vectors (which usually denote the links) and then constrain these points and vectors. In this formulation, the joints are encapsulated in the formulation itself and the constraint equations refer to the rigidity of the links. Hence, there are usually no constraint equations for the joints, making them unusable for RSA. In the next section, we show that natural coordinates can be made usable for RSA by introducing extra points and unit vectors. Natural coordinates offer the benefit that all the constraints are maximally quadratic[15, 16], while in absolute coordinates we have transcendental terms.

We illustrate the absolute and natural coordinates with a simple planar 4-bar mechanism, as shown in Figure 1.



(a) Natural Coordinates (b) Absolute Coordinates

Fig. 1. Planar 4-Bar Modelled with different Coordinates

3.1 Joint Reactions from Natural Coordinates

The constraint equations for 4-bar modeled using natural coordinates, as shown in Figure 2, are:

$$(x_1 - x_a)^2 + (y_1 - y_a)^2 = l_1^2$$
$$(x_1 - x_2)^2 + (y_1 - y_2)^2 = l_2^2$$
$$(x_2 - x_3)^2 + (y_2 - y_3)^2 = l_3^2$$

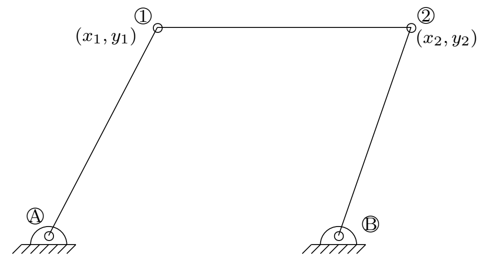


Fig. 2. 4-Bar Modelled with Natural Coordinates

In these equations, the first corresponds to the rigid-body constraint for the first link, the second one for the rigidity of second link and so on. The revolute joints in the mechanism are captured by the sharing of the "basic" points, hence no additional constraint equations are required for them. We propose a modified formulation of natural coordinates which help us write constraint equations for other kinds of joints. This formulation requires more "basic" points (and "unit vectors" for spatial mechanisms).

3.2 Joint-Augmented Natural Coordinate Formulation

Our modified natural coordinates formulation, which we call **joint-augmented natural coordinate formulation**, is shown in Figure 3. The key innovation here is that we have added extra geometric points to the links. While previously, for example, points ② and ③ of the second and third links were the same, now we have to add extra constraint equations to make them coincident. Owing to these extra constraint equations, we now have equations constraining the joints. Note that we have dismantled link-ends in the figure to show clearly how the points between different links are not shared as before, rather we share them using extra constraint equations.

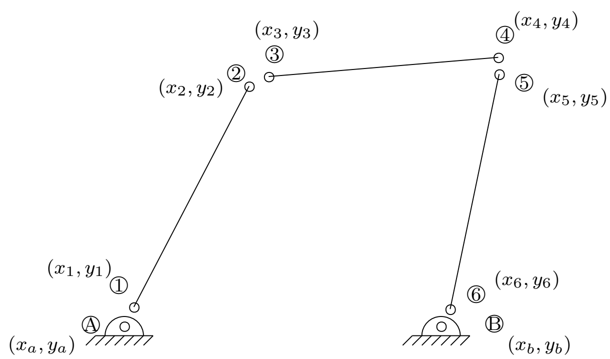


Fig. 3. 4-Bar Modelled with Modified Natural Coordinates

We now have extra constraints that correspond to the revolute joints. They are captured by 4 vector equations – hence a total 8 scalar equations – given as

$$\begin{aligned} \vec{r_A} &= \vec{r_1}, \qquad \vec{r_2} = \vec{r_3} \\ \vec{r_4} &= \vec{r_5}, \qquad \vec{r_6} = \vec{r_B} \end{aligned} \tag{7}$$

This makes natural coordinates usable for RSA.

4 RSA of a Planar Mechanism using Natural Coordinates

In this section, we will implement our RSA methodology on a planar over-contrained mechanism using the joint-augmented natural coordinates. This mechanism was analyzed in[10] using absolute coordinates. The mechanism is shown in Figure 4.

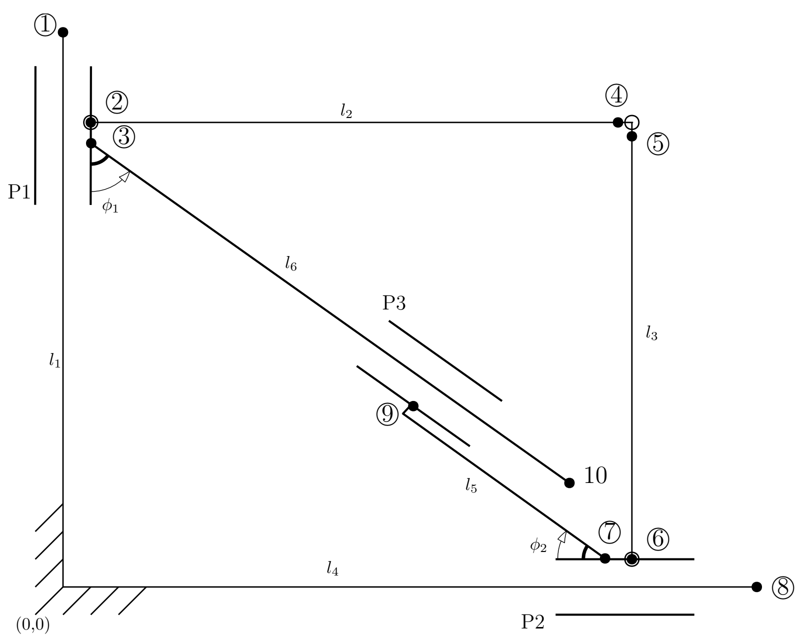


Fig. 4. A Planar over-constrained mechanism

4.1 Assumptions

We assume the following for this mechanism:

- All bodies are considered rigid with no flexibility,
- All joints are considered ideal, having only holonomic constraints, and
- No friction is present at the joint.

4.2 Modeling using Natural Coordinates

We follow the reference [4] to obtain the constraints with natural coordinates. The prismatic (P) joint constraints are of 2 types: one, collinear constraint (making cross product of the axes-defining vectors zero), and two, constraint for fixing the angle between the 2 connected bodies. Hence, for each prismatic joint, the constraint equations are

$$\text{P1:} \quad \left.\begin{array}{l} \vec{r_1} \times \vec{r_3} = 0 \\ (\vec{r_3} - \vec{r_{10}}).\vec{r_1} = l_1 l_6 \cos(\phi_1) \end{array}\right\} \quad \text{Prismatic Joint 1} \tag{8}$$

$$\text{P2:} \quad \left.\begin{array}{l} \vec{r_7} \times \vec{r_8} = 0 \\ (\vec{r_9} - \vec{r_7}).\vec{r_8} = l_5 l_4 \cos(\phi_2) \end{array}\right\} \quad \text{Prismatic Joint 2} \tag{9}$$

$$\text{P3:} \quad \left.\begin{array}{l} (\vec{r_{10}} - \vec{r_3}) \times (\vec{r_3} - \vec{r_9}) = 0 \\ (\vec{r_{10}} - \vec{r_3}).(\vec{r_9} - \vec{r_7}) = l_5 l_6 \end{array}\right\} \quad \text{Prismatic Joint 3} \tag{10}$$

For the revolute (R) joint, the points lying on the 2 bodies are co-incident and we get the constraint equations as:

$$\text{R1:} \quad \vec{r_2} = \vec{r_3} \} \quad \text{Revolute Joint 1} \tag{11}$$

$$\text{R2:} \quad \vec{r_4} = \vec{r_5} \} \quad \text{Revolute Joint 2} \tag{12}$$

$$\text{R3:} \quad \vec{r_7} = \vec{r_6} \} \quad \text{Revolute Joint 3} \tag{13}$$

The rigid-body and fixed (grounding) constraints are

$$\begin{array}{ll} \vec{r_1} = (0, l_1)^T & \vec{r_8} = (l_4, 0)^T \\ |\vec{r_4} - \vec{r_2}| = l_2 & |\vec{r_7} - \vec{r_9}| = l_5 \\ |\vec{r_5} - \vec{r_6}| = l_3 & |\vec{r_3} - \vec{r_{10}}| = l_6 \end{array} \tag{14}$$

4.3 Applying RSA Algorithm

The RSA Algorithm as discussed in section 2.2 was applied on this mechanism using Mathematica®. When using natural coordinates, the rank of Jacobian matrix is $r = rank(\Phi_q) = 19$ and the ranks r_i and r_{-i} are shown in the Table 1. For all the prismatic joints, $r_i + r_{-i} > r = 19$. Hence, for none of the prismatic joints, the reactions can be found uniquely. However, for all revolute joints $r_i + r_{-i} = r = 19$ and all the revolute joint reactions can be found uniquely.

When using absolute coordinates, as was done in [10], the rank r of the Jacobian matrix is 11. Here, for all prismatic joints, $r_i + r_{-i} > r = 11$ and for all revolute joints $r_i + r_{-i} = r = 11$. Hence, all revolute joint reactions can be found uniquely, but none of the prismatic joints can be. This is the same result as when using the natural coordinates. Results from both analyses are shown in Table 1 for comparison.

[1] whether the joint reaction can be found uniquely (Y) or not (N)

	Natural Coordinates				Absolute Coordinates			
i	$r_i = rank(\Phi_q^i)$	$r_{-i} = rank(\Phi_q^{-i})$	$r_i + r_{-i}$	Y/N[1]	$r_i = rank(\Phi_q^i)$	$r_{-i} = rank(\Phi_q^{-i})$	$r_i + r_{-i}$	Y/N
P1	2	18	20	N	2	10	12	N
P2	2	18	20	N	2	10	12	N
P3	2	18	20	N	2	10	12	N
R1	2	17	19	Y	2	9	11	Y
R2	2	17	19	Y	2	9	11	Y
R3	2	17	19	Y	2	9	11	Y

Table 1. Methodology-2 Results for the Planar Mechanism

4.4 Discussion

As can be seen from Table 1, both approaches of RSA analysis – using modified natural coordinates and using absolute coordinates (as done by[10]) yielded same results. Further, it can be observed that natural coordinates had larger matrices (with greater rank). However, this did not lead to increased computational complexity, as Jacobian matrix obtained using natural coordinates had no transcendental terms as seen when absolute coordinates are used. This led to a great increase in speed of the RSA algorithm when using natural coordinates. Three more mechanisms were analyzed apart from this planar over-constrained mechanism, including one spatial mechanism (Bennett mechanism), in which the computational advantage was found to be even more profound.

5 Conclusions

In this paper, we did a comparative study of various coordinate formulations which are usable for "Reaction Solvability Analysis (RSA)". It was found that only Absolute Coordinates are directly usable. However, a small modification in the Natural Coordinates was found to make them usable for RSA. Being able to use Natural Coordinates for RSA is good news as they offer simpler equations, making the RSA algorithm operate much faster on a given mechanism. This was demonstrated by performing the RSA using Natural Coordinates on a planar over-constrained mechanism. RSA yielded same result using both coordinates, while being faster when using Natural Coordinates.

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